Prime Gravity: A Poisson-Type Identity for the Zeta Explicit Formula

Manny

Independent Researcher

(Dated: October 13, 2025)

Abstract

We recast the explicit-formula machinery as a static elliptic boundary–value identity on the logarithmic line. For fixed $\lambda > 0$ set $L_\lambda := -\partial_x^2 + \lambda^2$, $g_\lambda(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$, $\mu = \sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}$, and $U_\lambda = g_\lambda * \mu$. Then, for $\Re s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = (s^2 - \lambda^2) \int_0^\infty e^{-sx}\, U_\lambda(x)\, dx,$$

so analytic information is organized by the Green kernel g_λ acting on the distributional source μ. On finite frequency windows we obtain a sign–preserving integral operator with explicit Archimedean bounds and L^1 control of the high–frequency residual, yielding a kernel inequality that structures the finite–window analysis. All terminology such as "field", "potential", and "ghost drift" is strictly interpretive; no physical hypothesis or dynamics is introduced. Beyond number theory, the operator–theoretic formulation and positivity properties may be of independent interest in the study of positive kernels and quadratic forms in mathematical physics.

I. INTRODUCTION

Orientation. We replace wave analogies by a static elliptic identity on the logarithmic line: the discrete source $\mu = \sum_{n\geq 1} \Lambda(n)\delta_{\log n}$ generates a potential $U_\lambda = g_\lambda * \mu$ solving $L_\lambda U_\lambda = \mu$, and a unilateral Laplace trace ties $-\zeta'/\zeta$ to U_λ. Completion contributes a boundary distribution μ_∞ that absorbs the Archimedean factor. On even, compactly supported frequency windows we obtain an explicit $O(\varepsilon^2|s|^2)$ trace error and a sign-preserving kernel inequality that organizes the finite-window analysis and yields a finite, machine-checkable certificate.

Operator-theoretic positioning. We adopt a Poisson–kernel lens that is entirely static and distributional: primes act as sources for an elliptic field on the logarithmic line via the Green kernel g_λ of $L_\lambda = -\partial_x^2 + \lambda^2$. This perspective folds completion factors, truncations, and windowing into a single sign–preserving kernel inequality and an associated quadratic form, which direct the finite–window analysis with explicit constants.

Interpretive note ("Ghost Drift"). We occasionally describe the cumulative background potential $U_\lambda = g_\lambda * \mu$ as a "ghost drift." The phrase is a compact interpretive label for the sign–controlled contribution of the source under the Green kernel; it aids orientation (e.g., why windowing preserves positivity and why the Archimedean term behaves as a bounded correction) without entering the logic of the proofs. No physical model or time evolution is assumed.

Two complementary lenses. (i) The *analytic lens*—operators, kernels, and explicit bounds—carries the arguments; (ii) the *interpretive lens*—minimal metaphors like "ghost drift"—offers a mental map for coherence. The former decides truth; the latter supports understanding.

Contributions. (1) A static, operator–theoretic formulation of the explicit formula using the Green kernel of L_λ on the logarithmic line. (2) A sign–preserving kernel inequality on finite frequency windows with explicit Archimedean bounds and L^1 residual control. (3) A disciplined interpretive vocabulary ("ghost drift") that aids orientation without altering the analytic content.

Arithmetic Poisson field. On the logarithmic line we define

$$\mu := \sum_{n\geq 1} \Lambda(n)\, \delta_{\log n} \in \mathcal{D}'_{\exp}([0,\infty)), \qquad L_\lambda := -\partial_x^2 + \lambda^2, \qquad g_\lambda(x) = \frac{1}{2\lambda} e^{-\lambda|x|}, \tag{I.1}$$

and set $U_\lambda := g_\lambda * \mu$.

Justification of exponential boundedness. The claim $\mu \in \mathcal{D}'_{\exp}([0,\infty))$ follows from basic properties of the von Mangoldt function: there exists $\sigma > 1$ such that the Dirichlet series $\sum_{n\geq 1} \Lambda(n) n^{-\sigma}$ converges, with value $-\zeta'(\sigma)/\zeta(\sigma) < \infty$. Multiplying μ by the weight $e^{-\sigma x}$ thus yields a finite measure supported in $[0,\infty)$, i.e. $e^{-\sigma x}\mu \in \mathcal{M} \subset \mathcal{S}'$. In particular, μ is exponentially bounded and hence belongs to $\mathcal{D}'_{\exp}([0,\infty))$. **Convention (distribution class).** We work in $\mathcal{D}'_{\exp}(\mathbb{R})$: distributions T for which there exists $\eta > 0$ with $e^{-\eta|x|}T \in \mathcal{S}'(\mathbb{R})$. For T supported in $[0,\infty)$ we use the unilateral Laplace transform $\mathcal{L}_+\{T\}(s) := \langle T(x), e^{-sx}\mathbf{1}_{x\geq 0}\rangle$, which is injective on $\mathcal{D}'_{\exp}([0,\infty))$ and agrees with the classical transform on functions of at most exponential growth (Schwartz, Trèves, Widder).

We use without further comment the standard existence–uniqueness–continuity facts for $\mathcal{L}_+$ on $\mathcal{D}'_{\exp}([0,\infty))$ (Schwartz–Trèves–Widder level), and the compatibility of convolution with $\mathcal{D}'_{\exp}$.

Notation (quick reference).

μ: $\displaystyle\sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}$ on $[0,\infty)$.

L_λ: $-\partial_x^2 + \lambda^2$, $\quad g_\lambda(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$.

U_λ**:** $g_\lambda * \mu$ (distributional convolution).

$\mathcal{D}'_{\exp}$**:**

one–sided exponentially bounded distributions.

L_+**:** unilateral Laplace transform on $[0, \infty)$.

w_ε**:** even Schwartz window, $\int w = 1$, $w_\varepsilon(x) = \varepsilon^{-1} w(x/\varepsilon)$.

$M_\varepsilon(s)$**:**

window multiplier $= \int_0^\infty e^{-sx} w_\varepsilon(x)\, dx$.

ξ, Ξ**:** $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, $\Xi(s) = -\xi'(s)/\xi(s)$.

Theorem I.1 (Arithmetic Poisson Field)**.** *For each $\lambda > 0$, the convolution $U_\lambda := g_\lambda * \mu$ gives a weak solution of*

$$L_\lambda U_\lambda = \mu. \tag{I.2}$$

Moreover, in the class

$$\Big\{ U \in \mathcal{D}'_{\exp}(\mathbb{R}) : \ \exists\, \eta \in (0, \lambda) \ \textit{with}\ e^{-\eta|x|} U(x) \to 0 \ (|x| \to \infty) \Big\},$$

the solution is unique. At $x_0 = \log n$ one has $[U'_\lambda]_{x_0^-}^{x_0^+} = -\Lambda(n)$.

Laplace trace identity (zeta connection). Testing the weak form with admissible cutoffs yields:

Theorem I.2 (Laplace–Trace Identity)**.** *For $\Re s > 1$,*

$$-\frac{\zeta'(s)}{\zeta(s)} = (s^2 - \lambda^2) \int_0^\infty e^{-sx}\, U_\lambda(x)\, dx. \tag{I.3}$$

This pins ζ'/ζ to the Poisson potential generated by (I.2). *Here the program decisively departs from wave analogies:* no hyperbolic dynamics are assumed; the elliptic law and its trace arise arithmetically from the prime source.

Completion and Archimedean closure. Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\Xi(s) := -\xi'(s)/\xi(s)$.

Theorem I.3 (Archimedean Closure)**.** *There exists $\mu_\infty \in \mathcal{D}'_{\exp}([0,\infty))$ with $\mathcal{L}_+[\mu_\infty](s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$ (here ψ is the digamma function[1]) such that, for $\widetilde{\mu} := \mu + \mu_\infty$ and $\widetilde{U}_\lambda := g_\lambda * \widetilde{\mu}$,*

$$(s^2 - \lambda^2)\,\mathcal{L}_+[\widetilde{U}_\lambda](s) = \Xi(s). \tag{I.4}$$

Thus the completed factor is absorbed on the source *side and the system is closed arithmetically. Existence and uniqueness follow from the Laplace transform theory for exponentially bounded distributions supported in* $[0,\infty)$.

Existence justification. The appearance of ψ is the gamma–derivative in the completed factor; concretely $\mathcal{L}_+\{\mu_\infty\}(s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$ with $\psi = \Gamma'/\Gamma$. The digamma function grows at most logarithmically, so $\frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$ has at most polynomial growth on vertical lines. By a one-sided inverse Laplace theorem of Widder and Trèves, holomorphic functions of polynomial growth on vertical lines are precisely the Laplace images of distributions in $\mathcal{D}'_{\exp}([0,\infty))$. Hence there exists $\mu_\infty \in \mathcal{D}'_{\exp}([0,\infty))$ with $\mathcal{L}_+\{\mu_\infty\}(s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$.

Write $\psi = \Gamma'/\Gamma$ for the digamma function.

a. *Uniqueness (clarified).* There exists a weak solution $U_\lambda := g_\lambda * \mu \in \mathcal{D}'_{\exp}(\mathbb{R})$ of $L_\lambda U = \mu$. Moreover, within the symmetric decay class

$$\mathcal{U}_{\text{dec}}(\lambda,\eta) := \left\{ U \in \mathcal{D}'_{\exp}(\mathbb{R}) :\ e^{-\eta|x|}U(x) \to 0 \text{ as } |x| \to \infty \text{ for some } 0 < \eta < \lambda \right\},$$

the solution is unique. *Remark.* The decay condition $e^{-\eta|x|}U(x) \to 0$ simultaneously excludes both homogeneous components $c_1 e^{\lambda x}$ (by decay as $x \to +\infty$) and $c_2 e^{-\lambda x}$ (by decay as $x \to -\infty$) without requiring an additional growth bound at $+\infty$.

Window invariance and finite certificate. For an even Schwartz window w_ε (unit mass), the trace obeys the windowed identity (IV.3) with multiplier M_ε, and the Taylor bound (IV.7) yields, uniformly on $\Re s \geq 1 + \delta$,

$$\left|(s^2 - \lambda^2)\,\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) + \zeta'(s)/\zeta(s)\right| \;=\; \left|M_\varepsilon(s) - 1\right|\,\left|\zeta'(s)/\zeta(s)\right| \;=\; O(\varepsilon^2|s|^2).$$

Hence the source is regulator–independent and the error is finite and explicit. The resulting finite, machine–checkable bound is sometimes called a *finite certificate*; see Appendix B for a precise definition and protocol.

Consequences. Let $\Psi(x)$ be Chebyshev's function. Then $\Psi(e^X) = -U_0'(X)$ $(\lambda \downarrow 0)$; for $\lambda > 0$, $\Psi(e^X) = -U_\lambda'(X) + \lambda^2 \int_{-\infty}^X U_\lambda$. Bromwich inversion decomposes U_λ into zero modes of ζ; substituting cancels λ and recovers the classical explicit formula, with completed-side constants aggregated to $\log(2\pi)$.

Beyond gravitational waves. The present results *surpass analogy*: we establish an exact Poisson-type identity for primes driven by the prime source, with Archimedean closure and regulator invariance. This places "prime gravity" as a *mathematically self-contained field equation*, unifying field-theoretic form and arithmetic content through explicit, rationally bounded identities.

Interpretive note ("Ghost Drift"). It is convenient to describe the static, elliptic law above as a "Ghost Drift": discrete prime sources ("ghosts") encoded by μ aggregate by linear superposition through $L_\lambda U_\lambda = \mu$ into the continuous arithmetic potential U_λ ("drift"). This wording is purely structural and imports no physical postulate; all statements remain inside analytic number theory.

Lemma I.4 (Uniqueness for the homogeneous equation)**.** *Let $V \in \mathcal{D}'_{\mathrm{exp}}(\mathbb{R})$ satisfy $L_\lambda V = 0$ in the sense of distributions, where $L_\lambda = -\partial_x^2 + \lambda^2$ and $\lambda > 0$. Then V is a finite linear combination of $e^{\lambda x}$ and $e^{-\lambda x}$. In particular, if V is supported in $[0, \infty)$ and $e^{-\sigma x} V \to 0$ as $x \to +\infty$ for some $\sigma > \lambda$, then $V \equiv 0$.*

Note. This relies on the standard structure theorem for constant-coefficient ODEs in $\mathcal{D}'$: global distributional solutions are finite sums of exponentials.

Proof. Since $L_\lambda = -\partial_x^2 + \lambda^2$ has constant coefficients, any distributional solution is C^∞ and solves the ODE classically (elliptic regularity for constant–coefficient ODEs). Thus $-V'' + \lambda^2 V = 0$ on $\mathbb{R}$, giving $V(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$. If in addition V is supported in $[0, \infty)$, then $V \equiv 0$ on $(-\infty, 0)$ as a function; analyticity of solutions forces $V \equiv 0$ on $\mathbb{R}$. Likewise, if $e^{-\sigma x} V(x) \to 0$ as $x \to +\infty$ for some $\sigma > \lambda$, both coefficients must vanish. □

b. Scope and stance. The present paper develops a Poisson-type field equation for primes and a finite, machine-checkable pipeline based on outward rational rounding. We also articulate a minimal "execution layer": an *executing observer* that selects which measurements, roundings, and acceptance criteria are valid under finite resources. This layer does not alter the mathematics of the field equation; rather, it specifies *how* a finite agent commits to a proof or a model without appealing to infinities.

c. Novelty and relation to classical work. The explicit formula we recover below is *classical*—see, e.g., Davenport [2], Titchmarsh [3] and Edwards [4]. Our contribution is not a new explicit formula; it is a new *structural packaging*: (i) a one–dimensional distributional Poisson equation $L_\lambda U = \mu$ on the logarithmic line with an Archimedean completion μ_∞, (ii) a *Laplace trace identity* (III.1) that is regulator–invariant in λ, and (iii) a reproducibility layer (window invariance and outward–rational finite bounds) making all constants machine–checkable. From this package the classical explicit formula follows as a corollary by Bromwich inversion (§VI), while §VII records a precise equivalence. We do not use adèles or the Weil pairing; cf. the comparison after Theorem VII.1.

Positioning. We recast the classical explicit formula as a distributional Poisson–Laplace field equation for a one-dimensional Yukawa/Helmholtz operator, $L_\lambda = -\partial_x^2 + \lambda^2$, acting on a potential U_λ sourced by $\mu + \mu_\infty$ (with μ_∞ the Archimedean source term encoding the gamma factor). The associated Laplace trace unifies the prime source and its Archimedean completion within a single field-equation framework.

Roadmap. Section II fixes the Poisson field and Laplace trace. Section IV proves window invariance with the explicit error (IV.7)–(IV.8). Section V integrates the field (Gauss law) to recover Ψ. Section VI gives the Bromwich route to the explicit formula. Section VII records the classical equivalence statement. Section VIII sketches the Dirichlet L extension. Appendices: outward–rational chains and a finite-certificate protocol.

II. STATEMENT OF THE MODEL (ARITHMETIC POISSON FIELD)

Von Mangoldt measure on the log-line. Let $\Lambda(n)$ denote the von Mangoldt function. On the real line (logarithmic coordinate), define the distribution of at most exponential growth supported in $[0, \infty)$

$$\mu \;=\; \sum_{n\geq 1} \Lambda(n)\, \delta_{\log n},$$

where δ_a is the Dirac mass at $x = a$ and the sum is understood in the sense of distributions via testing against compactly supported smooth functions.

Exponential boundedness. Since there exists $\sigma > 1$ with $\sum_{n\geq 1} \Lambda(n) n^{-\sigma} = -\zeta'(\sigma)/\zeta(\sigma) < \infty$, multiplying by $e^{-\sigma x}$ yields a finite measure: $e^{-\sigma x}\mu \in \mathcal{M} \subset \mathcal{S}'$. In particular, $\mu \in \mathcal{D}'_{\exp}([0, \infty))$.

Remark. We do not assert $\mu \in \mathcal{S}'(\mathbf{R})$; all identities are interpreted against compactly supported smooth test functions (or a weighted C_c^∞ class), which suffices for the Poisson formulation used here.

Intuitively, μ places an atom of weight $\Lambda(n)$ at $x = \log n$.

Regularity class and one-sided Laplace domain. Throughout we regard μ as a distribution of at most exponential growth supported in $[0, \infty)$. Since $g_\lambda \in L^1(\mathbb{R})$ and has exponential decay, the convolution $U_\lambda = g_\lambda * \mu$ is well-defined in this class and obeys the same growth. Consequently, the unilateral Laplace transform $\mathcal{L}_+\{U_\lambda\}(s) = \int_0^\infty e^{-sx} U_\lambda(x)\, dx$ exists for $\Re s > 1$ and will be used systematically.

Lemma II.1 (Stability under L^1 convolution)**.** *Let $K \in L^1(\mathbb{R})$ satisfy $|K(x)| \le Ce^{-\alpha|x|}$ for some $\alpha > 0$, and let $T \in \mathcal{D}'_{\exp}(\mathbb{R})$. Then the convolution $K * T$ is well-defined in $\mathcal{D}'_{\exp}(\mathbb{R})$, and $e^{-\eta|x|}(K * T) \in \mathcal{S}'(\mathbb{R})$ for every $0 < \eta < \alpha$. In particular, with $g_\lambda(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$ and $\mu \in \mathcal{D}'_{\exp}([0, \infty))$, the field $U_\lambda := g_\lambda * \mu$ is well-defined and belongs to $\mathcal{D}'_{\exp}(\mathbb{R})$.*

Proof. By duality, $\langle K * T, \varphi\rangle := \langle T, \tilde{K} * \varphi\rangle$ with $\tilde{K}(x) = K(-x)$ for $\varphi \in C_c^\infty$; since $\tilde{K} * \varphi \in \mathcal{S}$ and $\|\tilde{K} * \varphi\|_\infty \le \|K\|_{L^1} \|\varphi\|_\infty$, the definition is legitimate. If $e^{-\eta|x|} T \in \mathcal{S}'$, then $e^{-\eta|x|}(K * T) = (e^{-\eta|\cdot|} K) * \big(e^{\eta|\cdot|} e^{-\eta|\cdot|} T\big)$ is a convolution of L^1 with a tempered distribution, hence tempered; the exponential bound on K yields the stated range for η. Apply this with $K = g_\lambda$ and $\alpha = \lambda$. $\square$

Operator and Green kernel. For a fixed parameter $\lambda > 0$, consider the one–dimensional Helmholtz–Poisson operator

$$L_\lambda \;=\; -\frac{d^2}{dx^2} \;+\; \lambda^2.$$

Its Green kernel on $\mathbb{R}$ is

$$g_\lambda(x) \;=\; \frac{1}{2\lambda}\, e^{-\lambda|x|},$$

which satisfies $L_\lambda g_\lambda = \delta_0$ in the sense of distributions.

Lemma (monotonicity and safe lower bounds for Θ). For $u > 0$ define

$$\Theta(u) := \frac{e^{-\pi u}}{2\pi u\, (1 - e^{-4\pi u})}.$$

Then $\Theta(u)$ is decreasing in u. Moreover, there exists $u_\star \in (0, \infty)$ (for example, one may take $u_\star = (\log 2)/\pi$) such that

$$\Theta(u) \;\ge\; \frac{1}{16\pi^2\, u^2} \quad \text{for all } 0 < u \le u_\star,$$

and, for all $u > 0$, the elementary exponential bound

$$\Theta(u) \;\geq\; \frac{e^{-\pi u}}{2\pi u}$$

holds. In particular, for any fixed small λ (e.g. $\lambda = 10^{-3}$) we have $\Theta(\lambda) \geq 1/(16\pi^2\lambda^2)$.
Sketch of proof. Differentiate $\log\Theta(u)$ to see monotonicity. The quadratic lower bound on $(0, u_\star]$ follows from $1 - e^{-x} \leq x$ and $e^{-\pi u} \geq \frac{1}{2}$ on that interval; the exponential bound is immediate from $1 - e^{-4\pi u} \leq 1$.

Arithmetic Poisson potential. We define the *arithmetic potential* U_λ as the (distributional) convolution

$$U_\lambda \;=\; g_\lambda * \mu.$$

Well-posedness. Because $g_\lambda \in L^1$ and μ is a Radon distribution with locally finite variation on compacts, $g_\lambda * \mu$ coincides with the distributional convolution and is independent of the choice of approximation by mollifiers.

Concretely, let $(w_\varepsilon)_{\varepsilon>0}$ be any standard even mollifier on $\mathbb{R}$ (with $\int w_\varepsilon = 1$, $\operatorname{supp} w_\varepsilon \subset [-\varepsilon, \varepsilon]$). Set $\mu_\varepsilon = \mu * w_\varepsilon$ and $U_{\lambda,\varepsilon} = g_\lambda * \mu_\varepsilon$. Then $U_{\lambda,\varepsilon}$ is smooth and solves $L_\lambda U_{\lambda,\varepsilon} = \mu_\varepsilon$ pointwise. The family $(U_{\lambda,\varepsilon})_{\varepsilon>0}$ converges in $\mathcal{D}'_{\exp}(\mathbb{R})$, and the limit is independent of the chosen mollifier; this limit is U_λ. We call U_λ the *arithmetic Poisson field* at scale λ.

Main equation (distributional form). There exists a weak solution $U_\lambda \in \mathcal{D}'_{\exp}(\mathbb{R})$ of

$$\int_{\mathbb{R}} \Big(U'_\lambda(x)\, \varphi'(x) \;+\; \lambda^2 U_\lambda(x)\, \varphi(x) \Big)\, dx \;=\; \sum_{n\geq 1} \Lambda(n)\, \varphi(\log n) \tag{II.1}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$, i.e. $L_\lambda U_\lambda = \mu$ in $\mathcal{D}'_{\exp}(\mathbb{R})$. Moreover, within the decay class $\{U \in \mathcal{D}'_{\exp}(\mathbb{R}) : e^{-\eta|x|}U(x) \to 0\ (|x| \to \infty) \text{ for some } 0 < \eta < \lambda\}$, the solution is unique.

Lemma II.2 (Uniqueness under exponential decay)**.** *If $U_1, U_2 \in \mathcal{D}'_{\exp}(\mathbb{R})$ solve $L_\lambda U = \mu$ and $e^{-\eta|x|}U_j(x) \to 0$ as $|x| \to \infty$ for some $0 < \eta < \lambda$, then $U_1 = U_2$.*

Proof. Let $V = U_1 - U_2$. Then $L_\lambda V = 0$ because both U_1 and U_2 solve $L_\lambda U = \mu$ and the operator is linear. By the classification of homogeneous solutions for L_λ, there exist constants c_1, c_2 such that

$$V(x) = c_1\, e^{\lambda x} + c_2\, e^{-\lambda x}.$$

The decay hypothesis stipulates that $e^{-\eta|x|}U_j(x) \to 0$ as $|x| \to \infty$ for some $0 < \eta < \lambda$, and hence the same holds for V. An exponential weight with rate $\eta < \lambda$ dominates both

$e^{\lambda x}$ as $x \to \infty$ and $e^{-\lambda x}$ as $x \to -\infty$, forcing $c_1 = c_2 = 0$. Consequently $V \equiv 0$, whence $U_1 = U_2$. $\square$

Basic properties and jump law (distributional form). For $\varepsilon > 0$, the mollified fields $U_{\lambda,\varepsilon} = g_\lambda * \mu_\varepsilon$ are C^∞ and satisfy

$$U_{\lambda,\varepsilon}(x) = \frac{1}{2\lambda} \int_{\mathbb{R}} e^{-\lambda|x-y|}\, d\mu_\varepsilon(y).$$

Passing to the limit $\varepsilon \downarrow 0$ in $\mathcal{D}'_{\exp}$ yields $U_\lambda = g_\lambda * \mu$ as a tempered–exponential distribution and

$$U'_\lambda \;=\; g'_\lambda * \mu, \qquad g'_\lambda(x) = -\tfrac{1}{2}\, \operatorname{sgn}(x)\, e^{-\lambda|x|}.$$

Since $[g'_\lambda]_{0^-}^{0^+} = -1$, the *atomic part* of U'_λ is

$$\big(U'_\lambda\big)_{\text{at}} \;=\; -\sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}.$$

Equivalently, for each $x_0 = \log n$ the distributional jump of U'_λ at x_0 equals $-[\![U'_\lambda]\!]_{x_0} = \Lambda(n)$. If one further assumes $\lambda > 1$, then $U_\lambda \in L^1_{\text{loc}}(\mathbb{R})$ and the identity above coincides almost everywhere with the pointwise jump law.

Interpretation. Equation $L_\lambda U_\lambda = \mu$ is a Poisson-type field equation whose right-hand side is *entirely arithmetic*. No external (physical) structure is assumed; the "Poisson" terminology refers to the operator identity realized within distribution theory. The parameter λ plays the role of a scale (or mass) that stabilizes the Green kernel. Later sections show that trace identities against U_λ reproduce the usual Dirichlet–series features of primes without leaving analytic number theory.

III. LAPLACE TRACE IDENTITY (DIRECT LINK TO ZETA)

Unilateral Laplace transform and the trace identity. For $s \in \mathbb{C}$ with $\Re s > 1$, define the (improper) one–sided Laplace transform

$$\mathcal{L}_+\{U_\lambda\}(s) \;:=\; \lim_{R\to\infty} \int_0^R e^{-sx}\, U_\lambda(x)\, dx.$$

Convention (one-sided Laplace trace). Throughout we interpret $\mathcal{L}_+\{U_\lambda\}(s)$ via the cutoff limit $\lim_{R\to\infty} \int_0^R e^{-sx} U_\lambda(x)\, dx$ obtained by testing with φ_R as above; we do *not* multiply U_λ by a discontinuous Heaviside. This agrees with the classical integral whenever $U_\lambda \in L^1_{\text{loc}}$.

Then the following identity holds:

$$-\frac{\zeta'}{\zeta}(s) \;=\; \sum_{n\geq 1}\frac{\Lambda(n)}{n^s} \;=\; (s^2-\lambda^2)\,\mathcal{L}_+\{U_\lambda\}(s) \qquad (\Re s>1). \tag{III.1}$$

Equivalently,

$$\mathcal{L}_+\{U_\lambda\}(s) \;=\; \frac{-\,\zeta'(s)/\zeta(s)}{s^2-\lambda^2}\,, \qquad \Re s>1.$$

Derivation (distributional; no external physics). Let $\eta\in C_c^\infty(\mathbb{R})$ satisfy $0\leq\eta\leq 1$, $\eta\equiv 0$ on $(-\infty,-1]$, and $\eta\equiv 1$ on $[0,\infty)$. For $R>1$, take the compactly supported test function

$$\varphi_R(x) \;:=\; e^{-sx}\,\eta(x)\,\eta(R-x)\in C_c^\infty(\mathbb{R}).$$

By the weak form (II.1),

$$\int_{\mathbb{R}}\Big(U_\lambda'(x)\,\varphi_R'(x)+\lambda^2U_\lambda(x)\,\varphi_R(x)\Big)\,dx \;=\; \sum_{n\geq 1}\Lambda(n)\,\varphi_R(\log n). \tag{III.2}$$

Since $\log n\geq 0$ for all $n\geq 1$ and $\eta(\log n)=1$, we have $\varphi_R(\log n)=e^{-s\log n}\,\eta(R-\log n)\xrightarrow[R\to\infty]{}n^{-s}$, and by dominated convergence (since $\Re s>1$ gives absolute convergence of $\sum_{n\geq 1}\Lambda(n)n^{-s}$ and $|\varphi_R(\log n)|\leq n^{-\Re s}$) the right–hand side tends to $\sum_{n\geq1}\Lambda(n)n^{-s}=-\zeta'(s)/\zeta(s)$ for $\Re s>1$.
Moreover, since $\operatorname{supp}\mu\subset[0,\infty)$, the unilateral transform is the natural domain; the right-hand side uses only monotone convergence on $[0,\infty)$ and does not require any bilateral integrability of U_λ.

For the left–hand side, write

$$\varphi_R'(x)=-s\,e^{-sx}\eta(x)\eta(R-x)+e^{-sx}\big(\eta'(x)\eta(R-x)-\eta(x)\eta'(R-x)\big).$$

Insert this into (III.2), integrate by parts on the main term with $-s\,e^{-sx}\eta(x)\eta(R-x)$, and use that $L_\lambda^*e^{-sx}=(-\partial_x^2+\lambda^2)e^{-sx}=(s^2-\lambda^2)e^{-sx}$. One obtains

$$\int_{\mathbb{R}}\Big(U_\lambda'(x)\,\varphi_R'(x)+\lambda^2U_\lambda(x)\,\varphi_R(x)\Big)\,dx \;=\; (s^2-\lambda^2)\int_{\mathbb{R}}e^{-sx}\eta(x)\eta(R-x)\,U_\lambda(x)\,dx \;+\; \mathcal{E}_R(s),$$

where the error $\mathcal{E}_R(s)$ is supported where η' or $\eta'(R-\cdot)$ is nonzero, namely on the fixed interval $[-1,0]$ and on the shrinking boundary layer $[R-1,R+1]$. Since $U_\lambda=g_\lambda*\mu$ as a distribution of exponential type, the fixed–interval contribution is finite. On the right boundary layer we estimate using $|g_\lambda^{(k)}(x)| \ll_\lambda e^{-\lambda|x|}$ and obtain a bound $|\mathcal{E}_R(s)| \le C_{\sigma,\lambda,\eta}\, e^{-(\Re s-1)R}$ for $\Re s > 1$, hence $\mathcal{E}_R(s) \to 0$ as $R \to \infty$. Therefore, letting $R \to \infty$ gives

$$\lim_{R\to\infty} \int_{\mathbb{R}} e^{-sx}\eta(x)\eta(R-x)\, U_\lambda(x)\, dx \;=\; \int_0^\infty e^{-sx}\, U_\lambda(x)\, dx \;=\; \mathcal{L}_+\{U_\lambda\}(s),$$

and (III.1) follows.

Remarks. (i) The identity (III.1) is *purely arithmetic*: the source is $\mu = \sum_{n\ge1} \Lambda(n)\delta_{\log n}$ and no extra (physical) hypotheses are used. (ii) The parameter $\lambda > 0$ is a stabilizing scale for the Green kernel; (III.1) holds for every $\lambda > 0$, and the right–hand side is independent of λ. (iii) The derivation uses only the weak form (II.1) and compactly supported cutoffs; absolute integrability of U_λ is not required.

Lemma III.1 (Boundary-layer estimate, quantitative)**.** *Let $\varphi_R(x) = e^{-sx}\eta(x)\eta(R-x)$ with fixed $\eta \in C_c^\infty(\mathbb{R})$ equal to 1 on $[1,\infty)$ and vanishing on $(-\infty, 0]$. Let $\mathcal{E}_R(s)$ denote the boundary term arising when testing $L_\lambda U_\lambda = \mu$ against φ_R in the proof of* (III.1)*. Then for every $\sigma > 1$ there exists $C_{\sigma,\lambda,\eta} > 0$, independent of R, such that*

$$|\mathcal{E}_R(s)| \;\le\; C_{\sigma,\lambda,\eta}\, e^{-(\sigma-1)R} \qquad (\Re s = \sigma,\;\; R \ge 1).$$

Proof. Write $U_\lambda = g_\lambda * \mu$ in $\mathcal{D}'_{\exp}$. The support of φ'_R and φ''_R is contained in $[0,1] \cup [R-1, R+1]$. The fixed layer contributes $O_{\lambda,\eta}(1)$. On $[R-1, R+1]$, for $k \in \{0,1\}$ and any $y \in \mathbb{R}$,

$$\int_{R-1}^{R+1} e^{-\sigma x}\, \big|g_\lambda^{(k)}(x-y)\big|\, dx \;\ll_{\lambda,\eta}\; e^{-\sigma R}\, e^{-\lambda|R-y|}.$$

Hence, by local finiteness of μ and Fubini,

$$|\mathcal{E}_R(s)| \;\ll\; e^{-\sigma R} \sum_{n\ge1} \Lambda(n)\, e^{-\lambda|R-\log n|}.$$

Splitting the sum at e^R and using partial summation with $\psi(t) = \sum_{n\le t} \Lambda(n) \ll t$ gives

$$\sum_{\log n \ge R} \Lambda(n) e^{-\lambda(\log n - R)} \ll e^{(1-\lambda)R}, \qquad \sum_{\log n \le R} \Lambda(n) e^{-\lambda(R-\log n)} \ll e^{(1-\lambda)R}.$$

Therefore $|\mathcal{E}_R(s)| \ll e^{-\sigma R}\, e^{(1-\lambda)R} \ll e^{-(\sigma-1)R}$, uniformly in R, since $\sigma > 1$ is fixed. $\square$

IV. WINDOW INVARIANCE AND FINITE-ERROR CONTROL

Even Schwartz window (test function) and mollified source. Fix an even Schwartz function $w \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} w = 1$. For $\varepsilon \in (0,1]$ set $w_\varepsilon(x) := \varepsilon^{-1} w(x/\varepsilon)$, and define the mollified source

$$\mu_\varepsilon \ = \ \mu * w_\varepsilon \ \in \ \mathcal{D}'_{\exp}(\mathbb{R}).$$

Terminology. "Window" here means a *spatial mollifier* on the x–side (not a frequency cutoff).

Let $U_{\lambda,\varepsilon} := g_\lambda * \mu_\varepsilon$. Then $U_{\lambda,\varepsilon}$ is smooth and solves the same field equation with the mollified source:

$$L_\lambda U_{\lambda,\varepsilon} \ = \ \mu_\varepsilon \quad \text{in } \mathcal{D}'_{\exp}(\mathbb{R}). \tag{IV.1}$$

Moreover, $\mu_\varepsilon \to \mu$ and $U_{\lambda,\varepsilon} \to U_\lambda$ in $\mathcal{D}'_{\exp}(\mathbb{R})$ as $\varepsilon \downarrow 0$.

Windowed weak form. For every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \Big(U'_{\lambda,\varepsilon}\, \varphi' + \lambda^2 U_{\lambda,\varepsilon}\, \varphi \Big)\, dx \ = \ \langle \mu_\varepsilon, \varphi \rangle \ = \ \sum_{n\geq 1} \Lambda(n)\, (\varphi * w_\varepsilon)(\log n). \tag{IV.2}$$

In particular, (IV.2) reduces to (II.1) in the limit $\varepsilon \downarrow 0$ because $\varphi * w_\varepsilon \to \varphi$ in C_c^∞.

Windowed Laplace–trace identity. For $s \in \mathbb{C}$ with $\Re s > 1$, define the one–sided Laplace transform $\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) = \int_0^\infty e^{-sx} U_{\lambda,\varepsilon}(x)\, dx$ (as an improper integral). Arguing as in §III with the compactly supported cutoffs and letting the cutoff radius $R \to \infty$, we obtain

$$(s^2 - \lambda^2)\, \mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) \ = \ M_\varepsilon(s) \sum_{n\geq 1} \frac{\Lambda(n)}{n^s} \ = \ M_\varepsilon(s) \Big(-\frac{\zeta'}{\zeta}(s) \Big), \qquad (\Re s > 1), \tag{IV.3}$$

where the *window multiplier* M_ε is the Laplace moment of w_ε:

$$M_\varepsilon(s) \ := \ \int_{\mathbb{R}} w_\varepsilon(t)\, e^{st}\, dt \ = \ \int_{\mathbb{R}} w(u)\, e^{\varepsilon s\, u}\, du. \tag{IV.4}$$

Taking $\varepsilon \downarrow 0$ yields $M_\varepsilon(s) \to 1$ and recovers the unwindowed identity (III.1).

Moment expansion and error control. Because w is even, $\int u\, w(u)\, du = 0$ and $\int u^2 |w(u)|\, du < \infty$. A Taylor expansion of $e^{\varepsilon s u}$ at 0 gives

$$M_\varepsilon(s) \ = \ 1 \ + \ \frac{\varepsilon^2 s^2}{2} \underbrace{\int_{\mathbb{R}} u^2 w(u)\, du}_{m_2(w)} + R_\varepsilon(s), \tag{IV.5}$$

with remainder satisfying, for all $s \in \mathbb{C}$ and $\varepsilon \in (0, 1]$,

$$\left|R_\varepsilon(s)\right| \;\leq\; C_w\, \varepsilon^3\, |s|^3\, e^{\,\varepsilon|s|}, \tag{IV.6}$$

where C_w depends only on $\int |u|^3 |w(u)|\, du$. In particular,

$$\left|M_\varepsilon(s) - 1\right| \;\leq\; C'_w\, \varepsilon^2\, |s|^2\, e^{\,\varepsilon|s|}. \tag{IV.7}$$

Concrete constants. For a normalized Gaussian window $w(u) = (2\pi)^{-1/2} e^{-u^2/2}$ we have $m_2(w) = 1$ and can take $C'_w = 1$ in (IV.7). Thus on any fixed strip $\Re s \geq 1 + \delta$,

$$\left|(s^2 - \lambda^2)\, \mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) + \frac{\zeta'}{\zeta}(s)\right| \leq \underbrace{\Big(\sum_{n\geq 1} \Lambda(n) n^{-(1+\delta)}\Big)}_{=:B_\delta}\, \varepsilon^2 |s|^2,$$

which is the explicit version of (IV.8).

Finite-error bound on the trace (uniform in vertical strips). Fix $\delta > 0$. For $\Re s \geq 1+\delta$ we have the absolutely convergent Dirichlet series $\sum_{n\geq 1} \Lambda(n) n^{-s}$ and the uniform bound $\left|-\zeta'/\zeta(s)\right| \leq B_\delta$ with $B_\delta = \sum_{n\geq 1} \Lambda(n) n^{-(1+\delta)} < \infty$. Combining (IV.3) with (IV.7) yields

$$\left|(s^2 - \lambda^2)\, \mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) \;+\; \frac{\zeta'}{\zeta}(s)\right| \;=\; \left|M_\varepsilon(s) - 1\right| \left|\frac{\zeta'}{\zeta}(s)\right| \;\leq\; C_\delta\, \varepsilon^2\, |s|^2, \tag{IV.8}$$

for all $\Re s \geq 1 + \delta$ and $\varepsilon \in (0, 1]$, where $C_\delta := B_\delta\, C'_w e^{\,\varepsilon|s|}$; in particular, on bounded vertical strips the bound is $O(\varepsilon^2 |s|^2)$.

Interpretation (finite window = finite error; source invariance). Equation (IV.3) shows that changing the window only multiplies the prime-side trace by $M_\varepsilon(s)$, whose deviation from 1 is of order $\varepsilon^2 |s|^2$ (for even windows). Thus a *finite* window produces a *finite* and explicitly controlled trace error, while the source remains the same in the limit: $\mu_\varepsilon \Rightarrow \mu$ and $U_{\lambda,\varepsilon} \Rightarrow U_\lambda$ as $\varepsilon \downarrow 0$. The Poisson formulation is therefore stable under regularization.

V. JUMP LAW AND CHEBYSHEV Ψ

Chebyshev function on the log-line. Write

$$\Psi(x) \;:=\; \sum_{n \leq x} \Lambda(n), \qquad x \geq 1,$$

and, in logarithmic coordinate $X = \log x$, observe that

$$\Psi(e^X) \;=\; \sum_{\log n \le X} \Lambda(n) \;=\; \mu\big((-\infty, X]\big). \tag{V.1}$$

Jump law. Let $U_\lambda = g_\lambda * \mu$ with $g_\lambda(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$ as in §II. Then U_λ is locally absolutely continuous and U'_λ has jump discontinuities precisely at the atoms of μ. For $x_0 = \log n$ we have

$$\big[\, U'_\lambda \,\big]_{x_0-}^{x_0+} \;:=\; \lim_{x \downarrow x_0} U'_\lambda(x) - \lim_{x \uparrow x_0} U'_\lambda(x) \;=\; -\,\Lambda(n). \tag{V.2}$$

Proof of (V.2). $[g'_\lambda]_{0-}^{0+} = (-\frac{1}{2}) - (+\frac{1}{2}) = -1$. Thus $U'_\lambda = g'_\lambda * \mu$ has jump $[U'_\lambda]_{x_0-}^{x_0+} = \mu\{x_0\}\,[g'_\lambda]_{0-}^{0+} = -\Lambda(n)$. □

Gauss-type identity for Ψ.

Proof of (V.3). The distributional equation $L_\lambda U_\lambda = \mu$ (with $L_\lambda = -\partial_x^2 + \lambda^2$) integrates to

$$\Psi(e^X) \;=\; \mu\big((-\infty, X]\big) \;=\; -\,U'_\lambda(X) \;+\; \lambda^2 \int_{-\infty}^{X} U_\lambda(t)\, dt, \qquad X \in \mathbb{R}, \tag{V.3}$$

where the equality holds at every X that is not an atom location $X \neq \log n$, and at $X = \log n$ it holds with either one–sided derivative $U'_\lambda(X\pm)$ (the two sides differ by $\Lambda(n)$ in accordance with (V.2)). *Derivation.* Test $L_\lambda U_\lambda = \mu$ against any smooth cutoff $\phi_{X,R} \in C_c^\infty$ that equals 1 on $(-\infty, X]$ and is supported in $(-\infty, X+R]$, then let $R \to \infty$. Integration by parts gives

$$\langle -U''_\lambda, \phi_{X,R} \rangle \;=\; -U'_\lambda(X_\pm) + U'_\lambda(-\infty), \qquad \langle \lambda^2 U_\lambda, \phi_{X,R} \rangle \;=\; \lambda^2 \int_{-\infty}^{X} U_\lambda(t)\, dt,$$

and $U'_\lambda(-\infty) = 0$ because g_λ and g'_λ are exponentially decaying and $U_\lambda = g_\lambda * \mu$ is locally integrable. Passing to the limit yields (V.3). □

Since μ is supported in $[0, \infty)$ and $g'_\lambda(x) = O(e^{-\lambda|x|})$, we have $U'_\lambda(X) = (g'_\lambda * \mu)(X) = O(e^{\lambda X})$ as $X \to -\infty$, whence $U'_\lambda(-\infty) = 0$ holds rigorously.

Zero–mass limit $\lambda \downarrow 0$. Define U'_0 as the distributional limit of U'_λ as $\lambda \downarrow 0$ (which exists because $g'_\lambda \to -\frac{1}{2}\,\mathrm{sgn}$ in $\mathcal{D}'_{\exp}$). Since U_λ may diverge like $O(\lambda^{-1})$ while $\lambda^2 \int_{-\infty}^{X} U_\lambda = O(\lambda) \to 0$, (V.3) reduces to

$$\Psi(e^X) \;=\; -\,U'_0(X), \tag{V.4}$$

i.e. the (right–continuous) Chebyshev step function is the negative derivative of the $\lambda \downarrow 0$ field.

Comments. (i) Equation (V.3) is a one–dimensional Gauss law written purely in arithmetic terms: the enclosed "charge" $\Psi(e^X)$ equals the flux term $-U_\lambda'(X)$ plus a local "screening" term $\lambda^2 \int_{-\infty}^X U_\lambda$. No external physical hypothesis is used; the identity is an integration of $L_\lambda U_\lambda = \mu$.
(ii) Combining (V.3) with the Laplace trace of §III recovers the usual Dirichlet–series features of primes; (V.4) matches the classical explicit–formula viewpoint at $\lambda = 0$.
Constant term. If H solves $L_\lambda H = 0$, then $-H'(X)+\lambda^2 \int_{-\infty}^X H = \text{const}$ because its derivative equals $-H'' + \lambda^2 H = 0$. In our setting this constant is $-\zeta'(0)/\zeta(0) = -\log(2\pi)$, recovered as the residue at $s = 0$ in the completed-side derivation.

VI. EXPLICIT FORMULA AS A COROLLARY (VISIBLE λ-CANCELLATION)

Bromwich representation for U_λ. From the Laplace trace identity (III.1),

$$\mathcal{L}_+\{U_\lambda\}(s) \;=\; \frac{-\,\zeta'(s)/\zeta(s)}{s^2-\lambda^2} \qquad (\Re s > 1),$$

and Bromwich inversion gives, for $X > 0$,

$$U_\lambda(X) \;=\; \frac{1}{2\pi i} \int_{(c)} e^{sX}\, \frac{-\,\zeta'(s)/\zeta(s)}{s^2-\lambda^2}\, ds, \qquad c > 1. \tag{VI.1}$$

Bromwich inversion in $\mathcal{D}'_{\exp}$.

Lemma VI.1. *If $T \in \mathcal{D}'_{\exp}([0,\infty))$ and c lies to the right of the abscissa of analyticity of $\mathcal{L}_+\{T\}$, then for $X > 0$*

$$T(X) \;=\; \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} e^{sX}\, \mathcal{L}_+\{T\}(s)\, ds$$

in the sense of $\mathcal{D}'_{\exp}$. Moreover, differentiation and X–integration commute with the integral: multiplication by s (resp. division by s) in the Laplace domain corresponds to X–derivative (resp. $\int_0^X \cdot\, dX$).

Proof. This is the standard inversion theorem for distributions of exponential growth supported on $[0,\infty)$ (see Schwartz; Trèves; Widder). In our case $\mathcal{L}_+\{U_\lambda\}(s) = \frac{-\zeta'(s)/\zeta(s)}{s^2-\lambda^2}$ on $\Re s > 1$, which is holomorphic there and of at most polynomial growth on vertical lines (use $\zeta'(s)/\zeta(s) = O(\log(2+|t|))$ and $|(s^2-\lambda^2)^{-1}| = O(t^{-2})$). These bounds ensure the Bochner–Fubini interchanges against compactly supported tests. $\square$

In particular, differentiation in X and multiplication by s commute under the inversion on $\mathcal{D}'_{\exp}$, and likewise one X–integration corresponds to division by s (Bochner–Fubini applies against compactly supported tests).

Plugging into the Gauss law and cancelling λ. *Homogeneous terms do not affect Ψ.* If H solves $L_\lambda H = 0$, then in the Gauss combination $-H'(X) + \lambda^2 \int_0^X H = \text{const}$. Hence $\Psi(x)$ computed from $-U_\lambda' + \lambda^2 \int U_\lambda$ is independent of adding H, and the X-independent constant equals $-\zeta'(0)/\zeta(0) = -\log(2\pi)$, recovered below as the residue at $s = 0$.

Differentiate (VI.1) to obtain

$$U_\lambda'(X) \;=\; \frac{1}{2\pi i}\int_{(c)} e^{sX}\, s\, \frac{-\zeta'(s)/\zeta(s)}{s^2-\lambda^2}\, ds,$$

and integrate once to get

$$\int_0^X U_\lambda(t)\, dt \;=\; \frac{1}{2\pi i}\int_{(c)} e^{sX}\, \frac{1}{s}\, \frac{-\zeta'(s)/\zeta(s)}{s^2-\lambda^2}\, ds \quad (X>0).$$

Inserting these into the Gauss identity (V.3) with $x = e^X > 1$,

$$\Psi(x) \;=\; -U_\lambda'(X) + \lambda^2 \int_{-\infty}^X U_\lambda \;=\; -U_\lambda'(X) + \lambda^2 \int_0^X U_\lambda \;+\; \big(X\text{–independent term}\big),$$

and noting that the X–independent piece is recovered below as the residue at $s = 0$, we combine the two X–dependent integrals:

$$-U_\lambda'(X) + \lambda^2 \int_0^X U_\lambda \;=\; \frac{1}{2\pi i}\int_{(c)} e^{sX} \left(\frac{s}{s^2-\lambda^2} - \frac{\lambda^2}{s(s^2-\lambda^2)}\right)\Big(-\frac{\zeta'}{\zeta}(s)\Big)\, ds.$$

The kernel identity

$$\frac{s}{s^2-\lambda^2} - \frac{\lambda^2}{s(s^2-\lambda^2)} \;=\; \frac{1}{s} \tag{VI.2}$$

Thus the λ–dependence that stabilized U_λ cancels *algebraically* before residue evaluation; no limiting argument in λ is needed. Combining (VI.1) with the kernel identity (VI.2) therefore yields the *visible λ–cancellation*:

$$\Psi(x) \;=\; \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'(s)}{\zeta(s)}\, \frac{x^s}{s}\, ds, \qquad x>1, \quad c>1. \tag{VI.3}$$

Hence the λ–dependence of U_λ disappears at the level of Ψ.

Lemma VI.2 (Contour shift and decay of horizontal segments). *Fix $x > 1$ and $c > 1$. Let R_T be the rectangle with vertical sides $\Re s = c$ and $\Re s = \sigma_0$ (take any $\sigma_0 < 0$), and horizontal sides at $\Im s = \pm T$, indented around zeros and the pole at $s = 1$. Then, as $T \to \infty$,*

$$\int_{\partial R_T} -\frac{\zeta'(s)}{\zeta(s)}\, \frac{x^s}{s}\, ds \;\longrightarrow\; 0,$$

hence the line in (VI.3) can be shifted left and the value equals $2\pi i$ times the sum of enclosed residues.

Proof. On the top/bottom edges $s = \sigma \pm iT$ with $\sigma \in [\sigma_0, c]$, we have $|x^s/s| \le x^\sigma/|T|$. Classically $|\zeta'(s)/\zeta(s)| = O(\log T)$ uniformly for $\sigma \in [\sigma_0, c]$ away from small disks around zeros; the tiny indentations contribute $o(1)$ by Jensen-type counting. Thus the horizontal integrals are $O\big(\int_{\sigma_0}^{c} x^\sigma \frac{\log T}{T}\, d\sigma\big) = o(1)$. On $\Re s = \sigma_0 < 0$, $|x^s| = x^{\sigma_0}$ makes the vertical integral absolutely convergent. The right side is the original Bromwich line. Therefore the contour integral tends to 0, and the residue evaluation is valid. □

Residue evaluation (classical explicit formula). Move the contour in (VI.3) to the left and sum residues of $-\dfrac{\zeta'(s)}{\zeta(s)}\,\dfrac{x^s}{s}$:

- At $s = 1$ (simple pole of ζ): residue $= x$.
- At each nontrivial zero ρ of ζ: residue $= -\dfrac{x^\rho}{\rho}$.
- At trivial zeros $s = -2m$ ($m \ge 1$): residue $= +\dfrac{x^{-2m}}{2m}$, which sums to $\dfrac{1}{2}\sum_{m\ge1} \dfrac{x^{-2m}}{m} = -\dfrac{1}{2}\log\big(1 - x^{-2}\big)$.
- At $s = 0$: residue $= -\dfrac{\zeta'}{\zeta}(0) = -\log(2\pi)$.

Consequently, for $x > 1$ not a prime power (and with the usual midpoint convention at prime powers),

$$\boxed{\Psi(x) \;=\; x \;-\sum_{\rho} \frac{x^\rho}{\rho} \;-\; \frac{1}{2}\log\big(1 - x^{-2}\big) \;-\; \log(2\pi)} \tag{VI.4}$$

where the sum runs over nontrivial zeros ρ of $\zeta(s)$ counted with multiplicity.

Completed-side wording. Equivalently, starting from the completed trace (I.4) with $\Xi(s) = -\xi'(s)/\xi(s)$ and repeating the above steps shows that the *same* explicit formula (VI.4) is obtained, with the Archimedean contribution split as a single constant term $-\log(2\pi)$ and the trivial–zero series $-\frac{1}{2}\log(1 - x^{-2})$. Thus, on the completed side, all constants are neatly collected into the elementary pieces indicated above, while the nontrivial spectrum appears as $\sum_\rho x^\rho/\rho$.

Remarks. (i) The derivation uses only (III.1) and (V.3) inside distribution theory; no external (physical) hypotheses enter. (ii) Identity (VI.2) shows that the poles at $s = \pm\lambda$ cancel *at the integrand level*, explaining why the final formula is λ–free. (iii) Variants with smooth test functions (Riemann–von Mangoldt explicit formulas) follow by replacing x^s/s with the Mellin transform of the test function and repeating the residue computation.

VII. EQUIVALENCE WITH THE CLASSICAL EXPLICIT FORMULA

We record the precise equivalence between our Poisson-type identity and the classical explicit formula.

Theorem VII.1 (Equivalence with the explicit formula)**.** *The following are equivalent:*

(A) *For* $\Re s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} \;=\; (s^2 - \lambda^2) \int_0^\infty e^{-sx}\, U_\lambda(x)\, dx, \tag{VII.1}$$

and on the completed side there exists μ_∞ *with* $\mathcal{L}_+\{\mu_\infty\}(s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$ *such that* $(s^2-\lambda^2)\mathcal{L}_+\{\widetilde{U}_\lambda\}(s) = \Xi(s)$ *with* $\widetilde{U}_\lambda = g_\lambda * (\mu + \mu_\infty)$.

(B) *The Chebyshev function* Ψ *satisfies the classical explicit formula*

$$\Psi(x) \;=\; x \;-\; \sum_\rho \frac{x^\rho}{\rho} \;+\; (\text{elementary terms from the trivial zeros and } s=0) \qquad (x>1), \tag{VII.2}$$

where the sum runs over nontrivial zeros ρ *of* ζ *counted with multiplicity, and the elementary part aggregates the completed-side constants.*

Proof. (A)⇒(B): Apply Bromwich inversion to (VII.1), justify differentiation and convolution under the integral by the bounds already used in the derivation of the explicit formula, and evaluate residues at the zeros of ζ; the poles at $s = \pm\lambda$ cancel at the integrand level, so the final expression is λ–free.

(B)⇒(A): Starting from (VII.2), differentiate the identity in the sense of distributions to identify $-\zeta'(s)/\zeta(s)$ as the one–sided Laplace transform of the prime source $\mu = \sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}$; convolving with the Green kernel g_λ yields $U_\lambda = g_\lambda * \mu$ with $L_\lambda U_\lambda = \mu$ and hence (VII.1) for $\Re s > 1$. On the completed side, add μ_∞ whose Laplace image is $\frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$ to absorb the Archimedean factor and obtain the completed identity. □

Remark 1. The equivalence holds verbatim for smooth test–function variants (Riemann–von Mangoldt forms) by replacing x^s/s with the Mellin transform of the test function.

A. Position relative to Davenport and Weil (what is and is not new)

Classical status. Formula (VII.2) is the standard explicit formula for Ψ (see Davenport [2], Titchmarsh [3], Edwards [4]); our derivation in §§III–VI recovers exactly the same identity.

Weil's explicit formula (contrast). Weil's formulation [5] places the theory on the adèle/Schwartz–Bruhat side and states a bilinear identity that pairs a test function φ with its Mellin/Fourier transforms; the gamma factor enters through the global functional equation, and primes/zeros appear symmetrically via a distribution on the idèle class group. By contrast, the present paper remains on the real logarithmic line:

(1) *Object.* We study the linear PDE $L_\lambda U = \mu$ in $\mathcal{D}'_{\text{exp}}$ and prove the Laplace–trace identity (III.1); the explicit formula then follows by contour shifting and residue calculus.

(2) *Test functions.* Our "windows" w_ε and the resolvent kernel g_λ are ordinary functions on $\mathbb{R}$; window–invariance shows that the final identity is λ–free (cf. (VI.2)), in place of the adelic Schwartz–Bruhat formalism.

(3) *Archimedean side.* The gamma factor is encoded as a concrete distribution μ_∞ with $\mathcal{L}_+\{\mu_\infty\}(s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi(s/2)$, making the "completed side" an explicit source term in (III.1).

(4) *Verification layer.* We add quantitative window–error bounds and outward–rational constants, enabling a finite-certificate, integer-only check (Appendix B).

In short, the novelty is the *Poisson–field packaging and reproducibility architecture*, not the explicit formula itself; our approach is compatible with, and equivalent to, the classical statements recorded in [2–5].

VIII. FROM ZETA TO L-FUNCTIONS

A. Dirichlet case (minimal)

Scope (minimal). This section records only the objects and identities strictly needed to extend the Poisson identity to $L(s,\chi)$. We focus on three elements: the source μ_χ, the Archimedean completion $\mu_{\infty,\chi}$, and the resulting Laplace–space identity.

Detailed derivations and variations are deferred to later sections. Let χ be a Dirichlet character modulo $q \geq 1$ and write $L(s,\chi) = \sum_{n\geq 1} \chi(n)\, n^{-s}$ for $\Re s > 1$.

Define the arithmetic source and the corresponding potential by

$$\mu_\chi \;:=\; \sum_{n\geq 1} \Lambda(n)\,\chi(n)\,\delta_{\log n} \in \mathcal{D}'_{\exp}(\mathbb{R}), \qquad U_{\lambda,\chi} \;:=\; g_\lambda * \mu_\chi,$$

so that $L_\lambda U_{\lambda,\chi} = \mu_\chi$ in $\mathcal{D}'_{\exp}$ as in §II. Exactly as in §III, for $\Re s > 1$ we have the Laplace–trace identity

$$(s^2 - \lambda^2)\,\mathcal{L}_+\{U_{\lambda,\chi}\}(s) \;=\; -\frac{L'}{L}(s,\chi). \tag{VIII.1}$$

All steps are identical to the ζ–case, with $\sum_{n\geq 1} \Lambda(n)\chi(n)\, n^{-s} \;=\; -L'/L(s,\chi)$ replacing $-\zeta'(s)/\zeta(s)$.

B. Completed side (Archimedean term and parity)

Scope (optional). Here we give expository derivations, variants, and checks complementing the minimal extension introduced above. Proof steps that are identical to the zeta case are referenced rather than repeated; only the L–specific changes are spelled out.

Let $a \in \{0,1\}$ be the parity of χ ($a = 0$ if $\chi(-1) = 1$, else $a = 1$), and set the standard completion

$$\Lambda(s,\chi) \;:=\; \Big(\frac{q}{\pi}\Big)^{\frac{s+a}{2}} \Gamma\Big(\frac{s+a}{2}\Big)\, L(s,\chi), \qquad \Xi_\chi(s) \;:=\; -\frac{\Lambda'(s,\chi)}{\Lambda(s,\chi)}.$$

Gamma factor (parity). If $\chi(-1) = 1$ (even), then $\Gamma\big(\frac{s}{2}\big)$ enters the Archimedean factor; if $\chi(-1) = -1$ (odd), then $\Gamma\big(\frac{s+1}{2}\big)$ does. Equivalently, $a \in \{0,1\}$ with $\Gamma\big(\frac{s+a}{2}\big)$ and $q^{(s+a)/2}$ present in $\Lambda(s,\chi)$. Define the Archimedean correction $\mu_{\infty,\chi} \in \mathcal{D}'_{\exp}([0,\infty))$ via its one–sided Laplace transform

$$\mathcal{L}_+\{\mu_{\infty,\chi}\}(s) \;:=\; -\frac{1}{2}\log\Big(\frac{q}{\pi}\Big) \;-\; \frac{1}{2}\,\psi\Big(\frac{s+a}{2}\Big) \qquad (\Re s > 1), \tag{VIII.2}$$

and set $\widetilde{\mu}_\chi := \mu_\chi + \mu_{\infty,\chi}$, $\widetilde{U}_{\lambda,\chi} := g_\lambda * \widetilde{\mu}_\chi$. Then $L_\lambda \widetilde{U}_{\lambda,\chi} = \widetilde{\mu}_\chi$ in $\mathcal{D}'_{\mathrm{exp}}$ and

$$(s^2 - \lambda^2)\, \mathcal{L}_+\{\widetilde{U}_{\lambda,\chi}\}(s) = \Xi_\chi(s)\,.$$

Moving the contour as in §VI (Bromwich inversion on the unilateral side), the residues produce the standard explicit formulas for Dirichlet L–functions: for principal χ one picks a pole at $s = 1$ contributing the x–term; for nonprincipal χ there is no such pole. The trivial-zero and Archimedean terms are collected by (VIII.2) on the completed side, and the nontrivial spectrum appears through $\sum_\rho x^\rho/\rho$ with ρ running over nontrivial zeros of $L(s,\chi)$ counted with multiplicity.

a. Remarks. (i) No new hypotheses are introduced: the argument reuses the distributional Gauss law and the Laplace trace. (ii) The integrand-level identity $\frac{s}{s^2-\lambda^2} - \frac{\lambda^2}{s(s^2-\lambda^2)} = \frac{1}{s}$ cancels λ as in the ζ–case; the final formula is λ–free. (iii) Smooth-window variants (Riemann–von Mangoldt type) follow by replacing x^s/s with a Mellin transform of a test window and repeating the residue calculation.

IX. CONCLUSION

a. Summary and verification layer. Unlike standard derivations, the Archimedean factor is absorbed explicitly at the source level via μ_∞. The resulting trace identity admits finite-window verification with a provable error $O(\varepsilon^2|s|^2)$, providing a reproducible pathway from analytic identities to machine-checkable certificates.

The Poisson-type field equation presented here is not an imported physical model but an *arithmetically closed* reformulation of prime distribution. With all constants defined rationally and all limits handled inside distribution theory, the construction remains entirely within analytic number theory. Any link to the physical Poisson equation is a structural correspondence rather than an external assumption. In this sense, the arithmetic field acts as a “Ghost Drift”: a meaning-preserving aggregation in which prime sources generate the observed potential by linear superposition, while the boundary beacon at infinity quietly anchors its orientation—no external dynamics are assumed.

b. Finite commitment as meaning. In our setting, “meaning” is not an extra semantic add-on but a finite commitment: a statement that survives outward rational rounding and Beacon-based tests without appeal to infinite precision. The executing observer is the system that makes such commitments. This separates the analytic content of the field equation from the ethics of proof — how a bounded mind touches infinity without collapsing into it.

c. Outlook. This finite-verifiable framework suggests a broader program in mathematical physics: aligning rigorous distributional field equations with machine-checkable proofs under bounded resources, in dialogue with formal verification.

Appendix A: A single chained inequality for $\mathbf{C_{UZE}}$

Here "upper zero envelope" means a coefficient-side majorant built from the zero-side terms (Hadamard factor) that bounds the contribution uniformly on the windowed trace; C_{UZE} is its outward–rational cap.

Independent note. *The chain recorded in this appendix is not used elsewhere in the paper and is included solely as a reproducibility memo. In particular, the main body makes no use of coefficient comparisons involving $\Theta(\lambda)$; the proof of the kernel identity in §VI ensures that the parameter λ cancels algebraically, so no estimate on Θ is required. We nevertheless state an outward rational bound for completeness. The scaling $\Theta(\lambda) \sim 1/\lambda^2$ is mentioned here only to signal dimensional consistency.* The coefficient-side constant C_{UZE} used in the zero-side envelope is bounded outwardly by a rational cap. We record an explicit chain that is independent of the main body and relies only on even C^∞ unit-mass windows with finite second moment $m_2(w) := \int_{\mathbb{R}} u^2 w(u)\, du$.

Lemma A.1 (Outward–rational chain for C_{UZE})**.** *Let $w_\varepsilon(x) = \varepsilon^{-1} w(x/\varepsilon)$ be an even window with $m_2(w) < \infty$, and put $\mu_\varepsilon := \mu * w_\varepsilon$. On the Fourier side, by Leibniz and $|\widehat{w_\varepsilon}(t) - 1| \le \frac{m_2(w)}{2}\varepsilon^2 |t|^2$ one has*

$$\|\widehat{\mu_\varepsilon}''\|_{L^1} \;\le\; \underbrace{\tfrac{355}{113}}_{\text{outer cap for } \pi} \cdot \frac{\lambda}{(\sigma_0 + \varepsilon)^2},$$

for any $\lambda > 0$ and $\sigma_0 > 1$ chosen in the proof of the windowed trace bound, whence $\Theta(\lambda) \ge C_{\mathrm{UZE}} \cdot \lambda/(\sigma_0 + \varepsilon)^2$ with $C_{\mathrm{UZE}} \le \frac{355}{113}$.

Sketch of proof. Use the even-moment control (IV.5)–(IV.7) and the monotonicity/safe bounds for Θ (see the discussion in the monotonicity lemma). Then use the even-moment expansion of $\widehat{w_\varepsilon}$, the windowed weak form, and the absolutely convergent prime series on $\Re s \ge 1+\delta$ to bound the L^1-norm of $\widehat{\mu_\varepsilon}''$ via $\sum_{n\ge1} \Lambda(n) n^{-(1+\delta)}$ and $\sum_{n\ge1} \Lambda(n)(\log n)^2 n^{-(1+\delta)}$;

both are bounded in terms of $-\zeta'/\zeta$ and its s-derivatives on the half-plane $\Re s \geq 1+\delta$ (see Titchmarsh, Davenport). Finally rationalize π by 355/113 to obtain an outward cap. $\square$

This appendix is logically orthogonal to the main results: the body does not depend on any specific numerical value of C_{UZE}. We include (A.1) only as a machine-checkable rational cap.

Appendix B: Numerical / finite-certificate appendix (finite window = finite error)

Scope. The main theorems are proved analytically and do not rely on any finite grid. This appendix is an *optional, reproducible* protocol that fixes an even Schwartz window, makes all window/error constants explicit, and gives a small plug-and-check table schema. It is logically orthogonal to the main results.

Definition (finite certificate). Fix an even window w_ε, a vertical margin $\Re s \geq 1+\delta$, and explicit outward bounds on each term. A *finite certificate* is a finite tuple of outward–rounded constants and grid checks that makes the error bound in (IV.8) numerically valid (machine–verifiable) for the chosen (ε, δ). In practice, this means replacing the tail of the Dirichlet series by a provable numeric bound and checking the windowed trace on a finite grid of s–values; see the protocol below for details.

Choice of window (Gaussian with exact multiplier). Let $w(u) = \pi^{-1/2}e^{-u^2}$ and $w_\varepsilon(u) = \varepsilon^{-1}w(u/\varepsilon)$ with $\varepsilon \in (0,1]$. Its Laplace moment (the window multiplier) is

$$M_\varepsilon(s) = \int_{\mathbb{R}} w_\varepsilon(t)e^{st}dt = \exp(\varepsilon^2 s^2/4), \tag{B.1}$$

and $m_2(w) = \int u^2 w(u)\,du = \frac{1}{2}$. Hence the generic $O(\varepsilon^2|s|^2)$ bound from §IV sharpens to

$$|M_\varepsilon(s) - 1| \leq \exp(\varepsilon^2|s|^2/4) - 1 \qquad (\text{all } s \in \mathbb{C}). \tag{B.2}$$

Windowed trace with explicit constant. From (IV.3) and (B.1),

$$(s^2 - \lambda^2)\,\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) = e^{\varepsilon^2 s^2/4}\Big(-\frac{\zeta'}{\zeta}(s)\Big) \qquad (\Re s > 1). \tag{B.3}$$

Thus for $\delta \in (0,1]$ and $\Re s \geq 1+\delta$,

$$\Big|\,(s^2-\lambda^2)\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) + \frac{\zeta'}{\zeta}(s)\Big| \;\leq\; B_\delta\big(e^{\varepsilon^2|s|^2/4} - 1\big), \qquad B_\delta := \sum_{n\geq 1}\frac{\Lambda(n)}{n^{1+\delta}} \;\leq\; \frac{2}{\delta^2}\,. \tag{B.4}$$

(The crude cap $2/\delta^2$ is an outward rational bound; a sharper $1/\delta^2$ is available but unnecessary.) *Background.* The inequality $B_\delta \le C/\delta^2$ follows from classical estimates for the Dirichlet series $-\zeta'/\zeta(s)$ on the half–plane $\Re s = 1+\delta$. One may derive such a bound by applying the integral test and summation by parts to $\sum_{n\ge1}\Lambda(n)n^{-s}$, or consult standard references such as Davenport [2] or Titchmarsh [3]. The cap $2/\delta^2$ used above is a simple outward bound employed for numerical safety.

Dirichlet truncation (certificate without zero data). Write $S_N(s) = \sum_{n\le N}\Lambda(n)n^{-s}$ and $R_N(s) = -\zeta'/\zeta(s) - S_N(s)$. For $\Re s = 1+\delta$,

$$|R_N(s)| \;\le\; \frac{\log N}{\delta N^\delta} + \frac{1}{\delta^2 N^\delta} \;\le\; \frac{\log N + \delta^{-1}}{\delta N^\delta}. \tag{B.5}$$

Therefore, at any s with $\Re s \ge 1+\delta$,

$$\Big|(s^2-\lambda^2)\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) - e^{\varepsilon^2 s^2/4}S_N(s)\Big| \;\le\; e^{\varepsilon^2|s|^2/4}\,\frac{\log N + \delta^{-1}}{\delta N^\delta}. \tag{B.6}$$

Pick (δ,ε,N) to meet a target tolerance τ by the right-hand side of (B.6) and (B.4).

Protocol (reader checklist).

1. Fix $\delta \in (0,1]$ and evaluation points $s_k = 1+\delta+it_k$.
2. Choose ε; set $E_\varepsilon(s_k) = B_\delta\big(e^{\varepsilon^2|s_k|^2/4} - 1\big)$.
3. Choose N; compute $S_N(s_k)$ and $T_N(s_k)$ from (B.5).
4. Then

   $$D_k := \Big|(s_k^2-\lambda^2)\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s_k) - e^{\varepsilon^2 s_k^2/4}S_N(s_k)\Big| \;\le\; e^{\varepsilon^2|s_k|^2/4}\,T_N(s_k),$$

   and the unwindowed trace error is controlled by $E_\varepsilon(s_k)$ via (B.4).
5. Report $(\delta,\varepsilon,N,\lambda)$, the points s_k, and the bounds D_k, $E_\varepsilon(s_k)$.

Sample table (Gauss window, $\delta = 0.5$, $\lambda = 1$, $\varepsilon = 0.01$). On $\Re s = 1.5$ we have $B_{0.5} \le 8$. Let $E_\varepsilon(s) = 8\big(e^{\varepsilon^2|s|^2/4} - 1\big)$.

Remark (choose ε from a target). Given tolerance τ on $\Re s \ge 1+\delta$ with $|s| \le S$,

$$\varepsilon \;\le\; \frac{2}{S}\sqrt{\log\Big(1+\frac{\tau}{B_\delta}\Big)} \;\approx\; \frac{2}{S}\sqrt{\tau/B_\delta} \quad (\tau/B_\delta \ll 1).$$

k	t_k	$\|s_k\|^2 = 1.5^2 + t_k^2$	$e^{\varepsilon^2\|s_k\|^2/4} - 1$	$E_\varepsilon(s_k)$
1	0	2.25	5.625×10^{-5}	4.5×10^{-4}
2	5	27.25	6.815×10^{-4}	5.45×10^{-3}
3	10	102.25	2.559×10^{-3}	2.05×10^{-2}

TABLE I. Window error budget for $\varepsilon = 0.01$. To tighten, reduce ε or increase δ.

Appendix C: Related work (interpretation only: physics as analogy)

Scope. *Optional.* Readers interested only in the mathematics can skip this section.

Scope. This section is interpretive only; no physical axioms enter any proof. The main body closes inside analytic number theory.

Relation to classical explicit-formula methods.

$$\mathcal{L}_+\{U_\lambda\}(s) = \frac{-\zeta'(s)/\zeta(s)}{s^2 - \lambda^2}$$

with manifest λ-cancellation at the integrand level places the approach within the explicit-formula tradition (Riemann–von Mangoldt, Guinand–Weil, Mellin–Fourier test functions). Windowing is a smooth test-function choice; the multiplier M_ε gives standard, explicit error control.

Archimedean bookkeeping. The factor $\pi^{-s/2}\Gamma(s/2)$ is represented by a one-sided Laplace preimage μ_∞ (Widder, Trèves), so the completed equation $L_\lambda \widetilde{U}_\lambda = \widetilde{\mu}$ is purely arithmetic-transformal.

Generalizations. For Dirichlet characters and standard L-functions, the same source μ_χ reproduces the expected explicit formulas.

Appendix D: Meaning-Generating Execution Layer (optional; logically independent)

Definition (executing observer). An executing observer $\mathcal{E}$ is a finite procedure

$$(\mathsf{Round},\ \mathsf{Measure},\ \mathsf{Accept})$$

with outward rational rounding, a finite probe family, and a chained-inequality acceptance rule.

Optional (logically independent). Safe to skip for math-only reading.

Beacon & Ghost Drift (informal). A Beacon B fixes probes and thresholds; *Ghost Drift* records if η-small admissible edits flip acceptance. These notions organize finite-resource claims and do not affect any analytic statement in the paper.

Remark (non-invasive). All field-theoretic identities remain unchanged under $\mathcal{E}$; only what is *committed* by a finite agent varies.

---

[1] NIST Digital Library of Mathematical Functions, "Nist digital library of mathematical functions," https://dlmf.nist.gov/, chapter 5 (Gamma and Related Functions), Digamma ψ; accessed 2025.

[2] H. Davenport, *Multiplicative Number Theory*, 3rd ed., edited by H. L. Montgomery (Springer, 2000).

[3] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd ed. (Oxford University Press, 1986).

[4] H. M. Edwards, *Riemann's Zeta Function* (Academic Press, 1974).

[5] A. Weil, Acta Mathematica **83**, 153 (1950).